

## ON KESTEN'S MULTIVARIATE CHOQUET-DENY LEMMA

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ABSTRACT. Let  $d > 1$  and  $(\mathbf{A}_n)_{n \in \mathbb{N}}$  be a sequence of independent identically distributed random matrices with nonnegative entries and no zero column. This induces a Markov chain  $M_n = \mathbf{A}_n M_{n-1}$  on the cone  $\mathbb{R}_{\geq}^d \setminus \{0\} = \mathbb{S}_{\geq} \times \mathbb{R}_{>}$ . We study harmonic functions of this Markov chain. In particular, it is shown that all bounded harmonic functions in  $\mathcal{C}_b(\mathbb{S}_{\geq}) \otimes \mathcal{C}_b(\mathbb{R}_{>})$  are constant. The idea of the proof is originally due to Kesten [*Renewal theory for functionals of a Markov chain with general state space*. Ann. Prob. 2 (1974), 355 – 386], but is considerably shortened here.

## 1. INTRODUCTION

Let  $d > 1$ . The cone of  $d$ -vectors with nonnegative entries is denoted by  $\mathbb{R}_{\geq}^d = [0, \infty)^d$  and its intersection with the unit sphere is denoted by  $\mathbb{S}_{\geq} := \{x \in \mathbb{R}_{\geq}^d : |x| = 1\}$ , where  $|\cdot|$  denotes the euclidean norm on  $\mathbb{R}^d$ . Any matrix  $\mathbf{a} \in \mathcal{M}_+ := M(d \times d, \mathbb{R}_{\geq})$  with no zero column leaves  $V := \mathbb{R}_{\geq}^d \setminus \{0\}$  invariant, hence it acts on  $\mathbb{S}_{\geq}$  by

$$\mathbf{a} \cdot x := \frac{\mathbf{a}x}{|\mathbf{a}x|}, \quad x \in \mathbb{S}_{\geq}.$$

If  $\mu$  is a distribution on  $\mathcal{M}_+$  satisfying

$$(1.1) \quad \mu(\{\mathbf{a} : \mathbf{a} \text{ has a zero column}\}) = 0,$$

then  $V$  is  $\mu$ -a.s. invariant, and one can define a *Markov Random Walk*  $(X_n, S_n)_{n \in \mathbb{N}_0}$  on  $V \simeq \mathbb{S}_{\geq} \times \mathbb{R}$  by

$$X_n = \mathbf{A}_n \cdot X_{n-1}, \quad S_n - S_{n-1} = \log |\mathbf{A}_n X_{n-1}|$$

where  $(\mathbf{A}_n)_{n \in \mathbb{N}}$  is a sequence of independent identically distributed (iid) random matrices with distribution  $\mu$ . Writing  $M_n = X_n e^{V_n}$  this is nothing but a polar decomposition of the Markov chain  $(M_n)_{n \in \mathbb{N}_0}$  induced by the action of  $\mu$  on  $V$ .

The aim of this note is to study its bounded harmonic functions under some additional continuity condition. Besides being of interest in its own right, the absence of nontrivial bounded harmonic functions (usually called Choquet-Deny lemma) for this particular Markov chain played an important role in the proof of Kesten's renewal theorem [5] and was recently used in [7] to determine the set of fixed points of the multivariate distributional equation

$$Y \stackrel{d}{=} \mathbf{T}_1 Y_1 + \dots + \mathbf{T}_N Y_N,$$

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where  $N \geq 2$  is fixed,  $Y, Y_1, \dots, Y_N$  are iid  $\mathbb{R}^d$ -valued random variables and independent of the random matrices  $(\mathbf{T}_1, \dots, \mathbf{T}_N) \in \mathcal{M}_+^N$ . A prominent case is the deterministic choice  $\mathbf{T}_1 = \dots = \mathbf{T}_N = \text{diag}(N^{-1/\alpha}, \dots, N^{-1/\alpha})$ ,  $\alpha \in (0, 1]$ , where the fixed points are the multivariate  $\alpha$ -stable laws.

## 2. STATEMENT OF RESULTS

Writing  $\text{int}(A)$  for the topological interior of a set  $A$ , recall that by the Perron-Frobenius Theorem, any  $\mathbf{a} \in \text{int}(\mathcal{M}_+)$  possesses a unique largest eigenvalue  $\lambda_{\mathbf{a}} \in \mathbb{R}_{>}$  with corresponding eigenvector  $w_{\mathbf{a}} \in \text{int}(\mathbb{S}_{\geq})$ .

**Definition 2.1.** A subsemigroup  $\Gamma \subset \mathcal{M}_+$  is said to satisfy condition (C), if

- (1) no subspace  $W \subset \mathbb{R}^d$  with  $W \cap \mathbb{R}_{\geq}^d \neq \{0\}$  satisfies  $\Gamma W \subset W$  and
- (2)  $\Gamma \cap \text{int}(\mathcal{M}_+) \neq \emptyset$ .

Denote by  $[\text{supp } \mu]$  the smallest closed semigroup of  $\mathcal{M}_+$  generated by  $\text{supp } \mu$  and write  $\mathcal{C}_b(E)$  for the set of continuous bounded functions on the space  $E$ . Writing  $\mathbf{\Pi}_n = \mathbf{A}_n \dots \mathbf{A}_1$ , define probability measures  $\mathbb{P}_{u,t}$  on the path space of  $(X_n, S_n)_{n \in \mathbb{N}_0}$  by

$$\mathbb{P}_{x,t}((X_0, S_0, \dots, X_n, S_n) \in B) = \mathbb{P}((x, t, \dots, \mathbf{\Pi}_n \cdot x, \log |\mathbf{\Pi}_n u| + t) \in B)$$

for all  $n \in \mathbb{N}$  and measurable  $B$ .

**Theorem 2.2.** Let  $[\text{supp } \mu]$  satisfy (C) and let (1.1) hold. Assume that  $L \in \mathcal{C}_b(\mathbb{S}_{\geq} \times \mathbb{R})$  satisfies

- (a)  $L(x, s) = \mathbb{E}_x L(X_1, s - S_1)$  for all  $(x, s) \in \mathbb{S}_{\geq} \times \mathbb{R}$ , and
- (b) for all  $z \in \text{int}(\mathbb{S}_{\geq})$ ,

$$\limsup_{y \rightarrow z} \sup_{t \in \mathbb{R}} |L(y, t) - L(z, t)| = 0.$$

Then  $L$  is constant.

Each pair of functions  $f \in \mathcal{C}_b(\mathbb{S}_{\geq})$ ,  $h \in \mathcal{C}_b(\mathbb{R})$  defines a function  $f \otimes h \in \mathcal{C}_b(\mathbb{S}_{\geq} \times \mathbb{R})$  by  $(f \otimes h)(u, s) := f(u)h(s)$ . Write  $\mathcal{C}_b(\mathbb{S}_{\geq}) \otimes \mathcal{C}_b(\mathbb{R})$  for the set of all finite linear combinations of such functions (tensor product). Then the following corollary is obvious:

**Corollary 2.3.** Let  $[\text{supp } \mu]$  satisfy (C) and let (1.1) hold. If  $L \in \mathcal{C}_b(\mathbb{S}_{\geq}) \otimes \mathcal{C}_b(\mathbb{R})$  is harmonic for the Markov chain  $(X_n, S_n)_{n \in \mathbb{N}_0}$ , then  $L$  is constant.

The further organisation of the paper is as follows. At first, we repeat for the readers convenience important implications of (C), based on [3]. Then we turn to the proof of the main theorem. It will be assumed throughout that  $d > 1$ , that  $[\text{supp } \mu]$  satisfies (C) and that (1.1) holds.

## 3. IMPLICATIONS OF CONDITION (C)

Under each  $\mathbb{P}_x$ ,  $x \in \mathbb{S}_{\geq}$ ,  $(X_n)_{n \in \mathbb{N}_0}$  constitutes a Markov chain with transition operator  $P : \mathcal{C}_b(\mathbb{S}_{\geq}) \rightarrow \mathcal{C}_b(\mathbb{S}_{\geq})$  defined by

$$Pf(y) = \int f(\mathbf{a} \cdot y) \mu(d\mathbf{a}) = \mathbb{E}f(\mathbf{A}_1 \cdot y).$$

Abbreviating  $\Gamma = [\text{supp } \mu]$ , write

$$W(\Gamma) = \overline{\{w_{\mathbf{a}} : \mathbf{a} \in \Gamma \cap \text{int}(\mathcal{M}_+)\}}$$

for the closure of the set of normalized Perron-Frobenius eigenvectors, and

$$\Lambda(\Gamma) = \{\log \lambda_{\mathbf{a}} : \mathbf{a} \in \Gamma \cap \text{int}(\mathcal{M}_+)\}$$

for the logarithms of the corresponding Perron-Frobenius eigenvalues.

**Proposition 3.1** ([3, Propositions 3.1 & 3.2]). *The set  $\Lambda(\Gamma)$  generates a dense subgroup of  $\mathbb{R}$ . There is a unique  $P$ -stationary probability measure  $\nu$  on  $\mathbb{S}_{\geq}$ , and  $\text{supp } \nu = W(\Gamma)$ .*

Since  $\mathbb{S}_{\geq}$  is compact, the uniqueness of  $\nu$  implies the following ergodic theorem (see [1])

$$(3.1) \quad \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=1}^n f(X_k) = \int f(y) \nu(dy) \quad \mathbb{P}_x\text{-a.s.}$$

for all  $x \in \mathbb{S}_{\geq}$ ,  $f \in \mathcal{C}_b(\mathbb{S}_{\geq})$ .

The proposition above also implies the following “weak” aperiodicity property of  $(S_n)_{n \in \mathbb{N}_0}$ , which is an adaption of condition I.3 in [5]:

**Lemma 3.2.** *There exists a sequence  $(\zeta_i)_{i \in \mathbb{N}} \subset \mathbb{R}$  such that the group generated by  $(\zeta_i)_{i \in \mathbb{N}}$  is dense in  $\mathbb{R}$  and such that for each  $\zeta_i$  there exists  $z \in \text{int}(\mathbb{S}_{\geq})$  with the following properties:*

- (1)  $\nu(B_\varepsilon(z)) > 0$  for all  $\varepsilon > 0$ .
- (2) For all  $\delta > 0$  there is  $\varepsilon_\delta > 0$  such that for all  $\varepsilon \in (0, \varepsilon_\delta)$  there are  $m \in \mathbb{N}$  and  $\eta > 0$ , such that for  $B := B_\varepsilon(z)$ :

$$(3.2) \quad \mathbb{P}_x(X_m \in B, |V_m - \zeta_i| < \delta) \geq \eta \quad \text{for all } x \in B.$$

The first property together with (3.1) entails that  $B$  is a recurrent set for  $(X_n)_{n \in \mathbb{N}}$ . By a geometric trials argument (see e.g. [2, Problem 5.10]), it follows that for all  $\delta > 0$  and sufficiently small  $\varepsilon > 0$  there is  $m \in \mathbb{N}$  such that

$$(3.3) \quad \mathbb{P}_x(|X_n - z| < \varepsilon, |X_{n+m} - z| < \varepsilon, |S_n - (S_{n+m} - \zeta_i)| < \delta \text{ i.o.}) = 1$$

We repeat the short proof of Lemma 3.2 from [3, Prop. 5.5], for it clarifies the importance of Proposition 3.1 and we want to strengthen the result a bit.

*Proof.* By Prop. 3.1, the set  $\Lambda(\Gamma)$  generates a dense subgroup of  $\mathbb{R}$ , hence it contains a countable sequence  $(\zeta_i)$  which still generates a dense subgroup. Fix  $\zeta_i = \log \lambda_{\mathbf{a}}$ . Then  $\mathbf{a} \in \Gamma \cap \text{int}(\mathcal{M}_+)$ , hence

$$z := w_{\mathbf{a}} \in W(\Gamma) \cap \text{int}(\mathbb{S}_{\geq}).$$

Referring again to Prop. 3.1,  $z \in \text{supp } \nu$ , thus (1) follows.

Now fix  $\delta > 0$ . Then for all  $\varepsilon > 0$  sufficiently small,

$$\begin{aligned} \mathbf{a} \cdot B_\varepsilon(w_{\mathbf{a}}) &\subset B_{\varepsilon/2}(w_{\mathbf{a}}), \\ |\log \lambda_{\mathbf{a}} - \log |\mathbf{a}x|| &< \delta/2 \quad \text{for all } x \in B_\varepsilon(w_{\mathbf{a}}) \end{aligned}$$

Since  $\mathbf{a} \in [\text{supp } \mu]$ , there is  $m \in \mathbb{N}$  such that  $\mathbf{a} = \mathbf{a}_m \dots \mathbf{a}_1$ ,  $\mathbf{a}_j \in \text{supp } \mu$ ,  $1 \leq j \leq m$ , hence for all  $\gamma > 0$ ,

$$\mathbb{P}(\mathbf{A}_n \dots \mathbf{A}_1 \in B_\gamma(\mathbf{a})) = \eta_\gamma > 0.$$

If  $\gamma > 0$  is chosen sufficiently small, then for all  $\mathbf{a}' \in B_\gamma(\mathbf{a})$ ,

$$\begin{aligned} \mathbf{a}' \cdot B_\varepsilon(w_{\mathbf{a}}) &\subset B_\varepsilon(w_{\mathbf{a}}), \\ |\log \lambda_{\mathbf{a}} - \log |\mathbf{a}'x|| &< \delta \quad \text{for all } x \in B_\varepsilon(w_{\mathbf{a}}) \end{aligned}$$

Consequently, for all  $x \in B_\varepsilon(w_{\mathbf{a}})$ ,

$$\mathbb{P}(|\mathbf{\Pi}_n \cdot x - w_{\mathbf{a}}| < \varepsilon, |\log |\mathbf{\Pi}_n x| - \log \lambda_{\mathbf{a}}| < \delta) \geq \eta_\gamma > 0.$$

Recalling the definition of  $\mathbb{P}_x$ , this gives (3.2).  $\square$

#### 4. PROOF OF THE MAIN THEOREM

Let  $L \in \mathcal{C}_b(\mathbb{S}_\geq \times \mathbb{R})$ . For a compactly supported function  $h \in \mathcal{C}_b(\mathbb{R})$  define

$$L_h(x, s) = \int L(x, s + r) h(r) dr.$$

If for each such  $h$ ,  $L_h$  is constant, then the same holds true for  $L$  itself – this can be seen by choosing a sequence  $h_n$  of probability densities, such that  $h_n(r) dr$  converges weakly towards the dirac measure in 0.

**Lemma 4.1.** *Let  $L \in \mathcal{C}_b(\mathbb{S}_\geq \times \mathbb{R})$  satisfy properties (a), (b) of Theorem 2.2. Then for any compactly supported  $h \in \mathcal{C}_b(\mathbb{R})$ ,  $L_h$  still satisfies (a), (b) and moreover:*  
(c) *For all  $z \in \text{int}(\mathbb{S}_\geq)$ ,*

$$\lim_{y \rightarrow z} \lim_{\delta \downarrow 0} \sup_{|t-t'| < \delta} |L_h(z, t) - L_h(y, t')| = 0.$$

*Proof.* That (a) and (b) persist to hold for  $L_h$  is a simple consequence of Fubini's theorem resp. Fatou's lemma.

In order to prove (c), let  $|L| \leq C$ . Consider

$$\begin{aligned} & \lim_{\delta \rightarrow 0} \sup_{y \in \mathbb{S}_\geq} \sup_{|t-t'| < \delta} |L_h(y, t) - L_h(z, t')| \\ &= \lim_{\delta \rightarrow 0} \sup_{y \in \mathbb{S}_\geq} \sup_{|t-t'| < \delta} \left| \int L(y, t' + r) h(r - (t - t')) dr - \int L(y, t' + r) h(r) dr \right| \\ &\leq \lim_{\delta \rightarrow 0} \sup_{|t-t'| < \delta} C \int |h(r - (t - t')) - h(r)| dr = 0, \end{aligned}$$

where the uniform continuity of  $h$  was taken into account for the last line. Combine this with (b) to obtain for all  $x \in \text{int}(\mathbb{S}_\geq)$ ,

$$\begin{aligned} & \lim_{y \rightarrow z} \lim_{\delta \downarrow 0} \sup_{|t-t'| < \delta} |L_h(z, t) - L_h(y, t')| \\ &\leq \lim_{y \rightarrow z} \lim_{\delta \downarrow 0} \sup_{|t-t'| < \delta} \left[ |L_h(z, t) - L_h(y, t)| + |L_h(y, t) - L_h(y, t')| \right] \\ &\leq \lim_{y \rightarrow z} \sup_{t \in \mathbb{R}} |L_h(z, t) - L_h(y, t)| + \lim_{\delta \rightarrow 0} \sup_{y \in \mathbb{S}_\geq} \sup_{|t-t'| < \delta} |L_h(y, t) - L_h(y, t')| = 0. \end{aligned}$$

$\square$

Consequently, in order to proof Theorem 2.2, we may w.l.o.g. assume that properties (a) – (c) hold for  $L$ .

*Proof of Theorem 2.2.* The burden of the proof is to show that for all the  $\zeta_i$  of Lemma 3.2,

$$(4.1) \quad L(x, s) = L(x, s + \zeta_i) \quad \text{for all } (x, s) \in \mathbb{S}_\geq \times \mathbb{R}.$$

If this holds true, then for any  $\sigma = \sum_{i=1}^N c_i \zeta_i$  with  $c_i \in \mathbb{N}$ ,

$$L(x, s) = L(x, s + \sigma) \quad \text{for all } (x, s) \in \mathbb{S}_\geq \times \mathbb{R}.$$

But the set of  $\sigma$ 's is dense in  $\mathbb{R}$ , thus by the continuity of  $L$ ,

$$L(x, s) = L(x, 0) \quad \text{for all } (x, s) \in \mathbb{S}_{\geq} \times \mathbb{R}.$$

Hence  $L(x, s)$  reduces to a function  $\tilde{L}$  on  $\mathbb{S}_{\geq}$ , which is then bounded harmonic for the ergodic Markov chain  $(X_n)_{n \in \mathbb{N}_0}$  – see (3.1), thus  $\tilde{L}$  is constant.

Now we are going to prove (4.1). Considering (a),  $L(X_n, s - S_n)_{n \in \mathbb{N}_0}$  constitutes a bounded, hence a.s. convergent martingale under each  $\mathbb{P}_x$  with

$$(4.2) \quad L(x, s) = \mathbb{E}_x \lim_{n \rightarrow \infty} L(X_n, s - S_n) \quad \text{for all } (x, s) \in \mathbb{S}_{\geq} \times \mathbb{R}.$$

Fix any  $\zeta_i$  and the corresponding  $z \in \text{int}(\mathbb{S}_{\geq})$ , defined in Lemma 3.2. Referring to (c), for all  $\xi > 0$ , there are  $\delta, \varepsilon > 0$  such that

$$\sup_{u, y \in B_{\varepsilon}(z)} \sup_{|t - t'| < \delta} |L(u, t) - L(y, t')| < \xi.$$

Combining this with (3.3), we infer that for all  $s \in \mathbb{R}$ ,

$$\mathbb{P}_x (|L(X_n, s - V_n) - L(X_{n+m}, s + \zeta_i - V_{n+m})| < \xi \text{ i.o.}) = 1.$$

Hence for all  $(x, s) \in \mathbb{S}_{\geq} \times \mathbb{R}$ ,

$$\lim_{n \rightarrow \infty} L(X_n, s - V_n) = \lim_{n \rightarrow \infty} L(X_n, s + \zeta_i - V_n) \quad \mathbb{P}_x\text{-a.s.}$$

and consequently, using (4.2), it follows for all  $(x, s) \in \mathbb{S}_{\geq} \times \mathbb{R}$

$$L(x, s) = \mathbb{E}_x \lim_{n \rightarrow \infty} L(X_n, s - V_n) = \mathbb{E}_x \lim_{n \rightarrow \infty} L(X_n, s + \zeta_i - V_n) = L(x, s + \zeta_i).$$

□

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